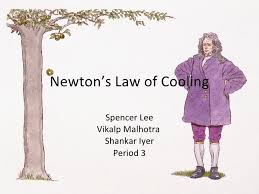


appklication of integrals

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NEWTONS LAW OF COOLING



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**Newton's Law of Cooling**

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| Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature (i.e. the temperature of its surroundings). |

Newton's Law makes a statement about an **instantaneous** rate of change of the temperature. We will see that when we translate this verbal statement into a differential equation, we arrive at a differential equation. The **solution** to this equation will then be a function that tracks the complete record of the temperature over time. Newton's Law would enable us to solve the following problem.

**The Big Pot of Soup**As part of his summer job at a resturant, Jim learned to cook up a big pot of soup late at night, just before closing time, so that there would be plenty of soup to feed customers the next day. He also found out that, while refrigeration was essential to preserve the soup overnight, the soup was too hot to be put directly into the fridge when it was ready. (The soup had just boiled at 100 degrees C, and the fridge was not powerful enough to accomodate a big pot of soup if it was any warmer than 20 degrees C). Jim discovered that by cooling the pot in a sink full of cold water, (kept running, so that its temperature was roughly constant at 5 degrees C) and stirring occasionally, he could bring tht temperature of the soup to 60 degrees C in ten minutes. How long before closing time should the soup be ready so that Jim could put it in the fridge and leave on time ?

**Solution:** Let us summarize the information briefly and define notation for this problem.  
Let

|  |
| --- |
| $ T(t) $  = Temperature of the soup at time t (in min). |
| $ T(0)=T_o $  = Initial Temperature of the soup =100 deg. |
| $ T_a $  = Ambient temperature (temp of water in sink) = 5 deg . |

**Given:** The rate of change of the temperature $ dT/dt $  , is (by Newton's Law of Cooling) proportional to the difference between the temperature of the soup $ T(t) $  and the ambient temperature $ T_a $  This means that:


\[ 
\frac{dT}{dt} {\rm~ is~ proportional~ to~} (T - T_a). 
\] 
 

Here a bit of care is needed: Clearly if the soup is hotter than the water in the sink $ T(t)-T_a > 0  $  , then the soup is **cooling down** which means that the derivative $ dT/dt $  should be **negative**. (Remember the connection between a decreasing function and the sign of the ). This means that the equation we need has to have the following sign pattern:


\[ 
\frac{dT}{dt} = -k (T - T_a). 
\] 
 

where $ k $  is a positive constant.  
This equation is another example of The independent variable is $ t $  for time, the function we want to find is $ T(t) $  , and the quantities $ T_a, k $  are constants. In fact, from Jim's measurements, we know that $ T_a=5 $  , but we still don't know what value to put in for the constant $ k $  . We will discuss this further below.

The equation we arrived at above looks different from the ones we have just investigated, but as we shall soon see, the difference is rather superficial. Indeed, by defining a new variable, we will show that the equation is really completely related to the exponential decay seen previously. To see this, define

|  |
| --- |
| $ y(t) =T(t)-T_a $  = Temperature difference between soup and water in sink at time t. |
| $ y_o = T(0) -T_a = T_o-T_a  $  = Initial temperature difference at time t=0 |

Note that if we take a derivative of $ y(t) $  , and use the Newton's law of cooling, we arrive at


\[ 
\frac{dy}{dt} = \frac{d}{dt}(T(t)-T_a)=  \frac{dT}{dt}-\frac{dT_a}{dt} =  \frac{dT}{dt}= -k (T-T_a) =-k y 
\] 
 

(We have used the fact that $ T_a  $  is constant to eliminate its derivative, and we plugged in $ y $  for $ (T-T_a) $  in the last step.) What a nice surprize ! By defining this new variable, we have arrived once more at the familiar equation

![
\[ 
\frac{dy}{dt} = -k y 
\] 
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whose solution is well known to us, namely:

![
\[ 
y(t) = y_o e^{-kt} 
\] 
 ](data:image/gif;base64,R0lGODlhXwAZAMIAAAAAAI+Pj19fXy8vL7+/vwAAAAAAAAAAACH5BAEAAAQALAAAAABfABkAQAP+SLrc/jDKSatdoo3Mxv0gBBBbQJoPYA6eIqCEkLULHTYq6QVjPKOym3BILBoZsEbyyGT0MEuToCXoPRUbR5AopV4xnClTltVGAsshmdNsu9+OwIvLViRtcKIsw/lixQo5NlMAWy5pIRs0OQ8bgBBoBHJ2dU4ojFc8D4Zge56cloGIfkdsjFgTk0amUn15r7CxsrO0tUN4oBSRjbYhkzJyhYgQuIPDtZ/AOjC5EqCngb0Wv3MjlZFPVg2PYFrJnnEm2SIYTWVbX9AulULn7ASFyk3j0RhUDHNG9AGK4khNcvYsULXOAakbAYNscfduSI8yUCTgcfjHTrxjsxo2lMYEsWObBAA7)

We can use this result to conclude (by plugging in $ y=(T-T_a) $  and $ (T_o-T_a) $  ) that


\[ 
T(t)- T_a = (T_o-T_a) e^{-kt} 
\] 
 

It follows that


\[ 
T(t)= T_a + (T_o-T_a) e^{-kt} 
\] 
 

REAL TIME PROBLEM

From the information in the problem, we know that

$ T(0)=T_o = 100, T_a = 5, T_o-T_a = 95  $ 

so that,


\[ 
T(t)= 5 + 95 e^{-kt} 
\] 
 

We also know that after 10 minutes, the soup cools to 60 degrees, so that $ t=10, T(10)= 60  $  . Plugging into the last equation, we find that


\[ 
60= T(10)= 5 + 95 e^{-10 k } 
\] 
 

Rearranging,


\[ 
60= 5 + 95 e^{-10 k }, ~~~~~~~~55=95 e^{-10 k },~~~~~~~~\frac{55}{95} = e^{-10 k }, 
~~~~~~~~~\frac{95}{55}= e^{10 k } 
\] 
 

(The steps are much the same as in our previous work in the example on radioactive decay. In the last step we took a reciprocal of both sides of the equation. This just makes all the quantities come out to be positive in the next step, so it is done for convenience, though it is not an essential step). We have found that

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\[ 
e^{10 k }= 1.73 
\] 
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Taking the natural logarithm of both sides, and solving for $ k $  , we find that


\[ 
\ln(1.73)= \ln(e^{10 k }),~~~~~~~~~~ \ln(1.73)= 10 k 
\] 
 

Thus,


\[ 
k =\frac{\ln(1.73)}{10}=\frac{0.54}{10}=0.054 
{\rm ~per~min} 
\] 
 

So we see that the constant which governs the rate of cooling is $ k=0.054 $  per minute. Now we can specify the solution fully, since all constants have been determined from the information in the problem. The prediction is that the temperature of the pot of soup at time t will be

|  |  |
| --- | --- |
| \[  T(t)= 5 + 95 e^{-0.054t}  \] |  |

